Matter Waves

Chapter 5

$$p = \frac{h}{\lambda}$$

De Broglie pilot waves

Electromagnetic waves are associated with quanta - particles called photons.

Turning this fact on its head, Louis de Broglie made a bold guess :

Matter particles should have wave properties.

De Broglie pilot waves

What is the wavelength and frequency of a matter wave? Extrapolate from Planck's result:

Photons:

$$f = \frac{E}{h} \quad \lambda = \frac{h}{p}$$
$$E = pc$$

Matter particles:
same except relationship between E and p

$$f = \frac{E}{h} \quad \lambda = \frac{h}{p}$$
$$E = \sqrt{(mc^2)^2 + (pc)^2}$$

De Broglie pilot waves

de Broglie wavelength example:

What is the wavelength of an electron of kinetic energy 100 eV?

De Broglie pilot waves

de Broglie wavelength example:

What is the wavelength of an electron of kinetic energy 100 eV?

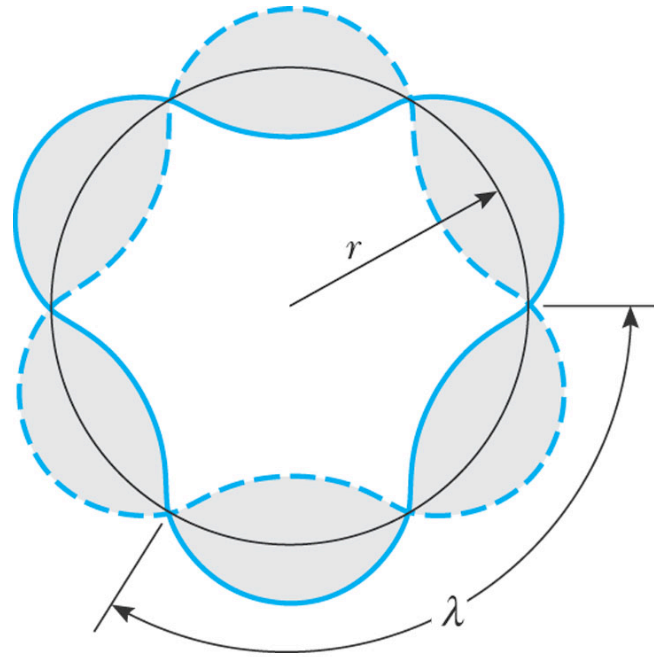
$$
\begin{aligned}
&p = \sqrt{2m_e E} \\
&= \sqrt{2(9.1e-31\ kg)(100\ eV)(1.6e-19)(J/eV)} \\
&= 5.4e-24\ kg-m/s \\
&\lambda = h/p \\
&= 6.6e-34\ J-s/5.4e-24\ kg-m/s = 1.2e-10m
\end{aligned}
$$

Note: this is comparable to atom radii.

Bohr atom and pilot waves

How this idea might explain Bohr's quantization condition: Suppose standing waves at radius r:

$$n\lambda = 2\pi r \quad \lambda = h/p$$
$$\Rightarrow n\frac{h}{2\pi} = pr$$
$$\Rightarrow n\hbar = L$$



(This is "hand waving.")

Waves

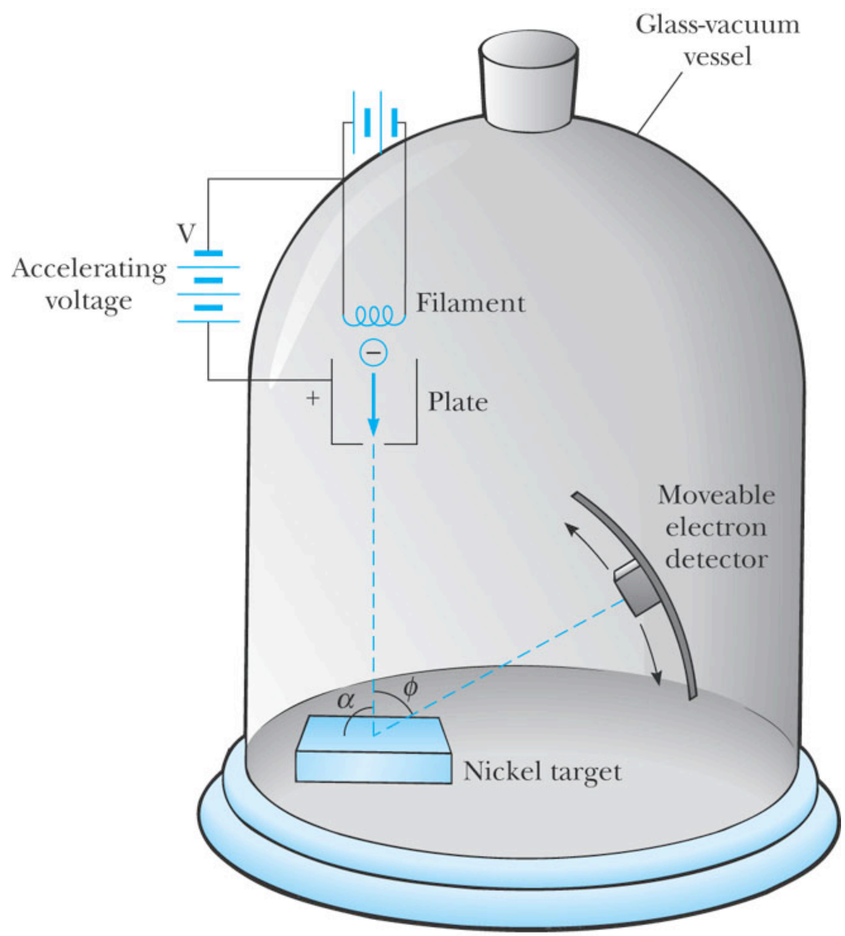
How can one see the wave properties of matter?

Look for superposition of waves in interference and diffraction.

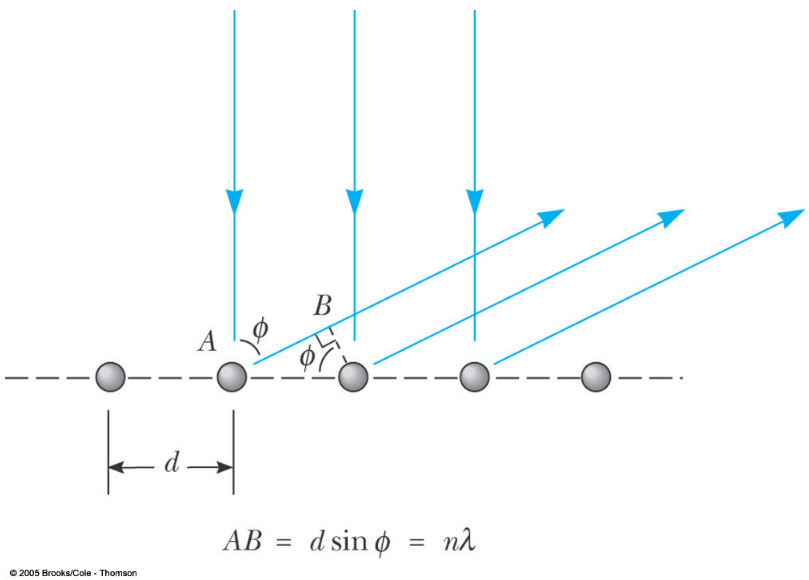
Davisson-Germer Experiment

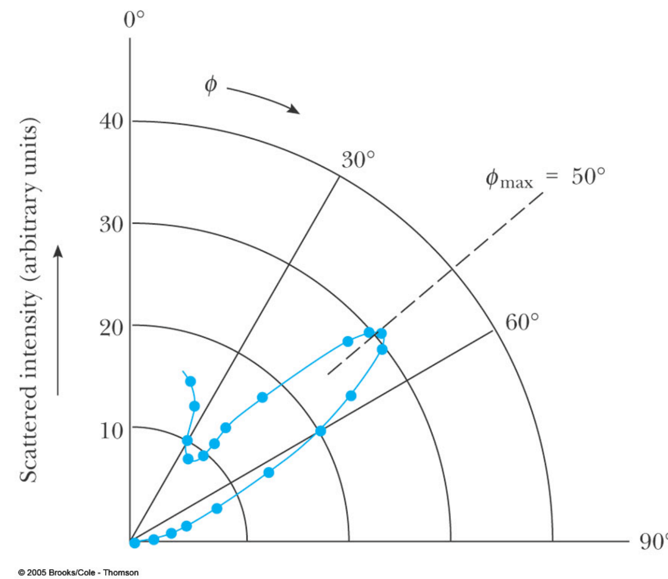
Direct evidence for wave nature of matter was first seen in electron diffraction at Bell labs in 1927.

Electrons scattered from a Ni surface exhibited an odd pattern.



Davisson Germer Experiment





A plot of scattering intensity versus angle for 54-eV electrons. The separation d= 0.215 nm was known from x-ray scattering.

$$d\sin\phi = (0.215\ nm)\sin(50\ deg) = 0.165\ nm$$

$$\lambda = h/p = h/\sqrt{2m_e E} = 0.167\ nm$$

More generally..

De Broglie’s idea applies not just to electrons!

All matter particles exhibit wave properties!

-neutrons
-protons
-electrons

-muons, quarks,…

-composites such as atoms

Quantum Mechanics
applies to everything!

Thermal neutrons

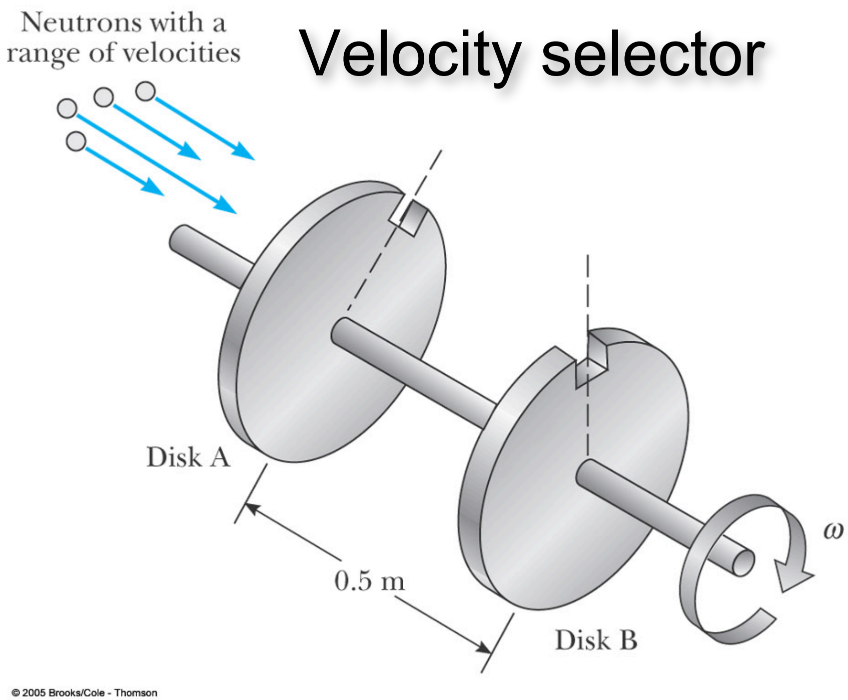
What energy neutron has a wavelength of 1 Angstrom?

Thermal neutrons

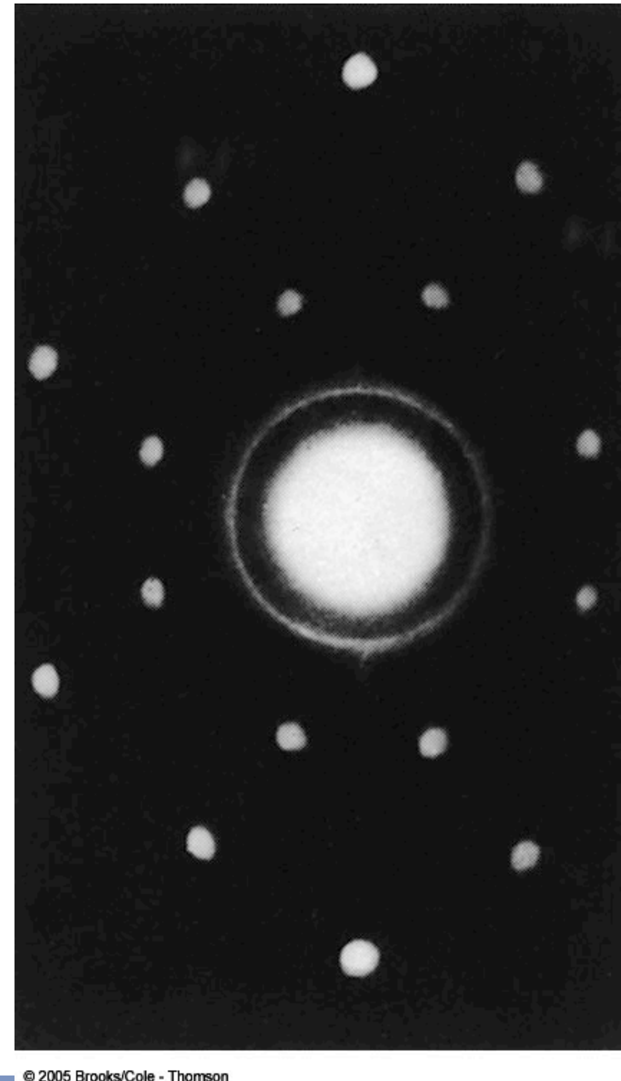
What energy neutron has a wavelength of 1 Angstrom?

The momentum is $p=[2mE]^{1/2}$. A 100 eV electron has this wavelength. The neutron weighs 2000 times as much as the electron so for the same wavelength must have 2000 times less energy or 50 meV. This is comparable to kT at room temperature.

Neutron diffraction



Velocity selector

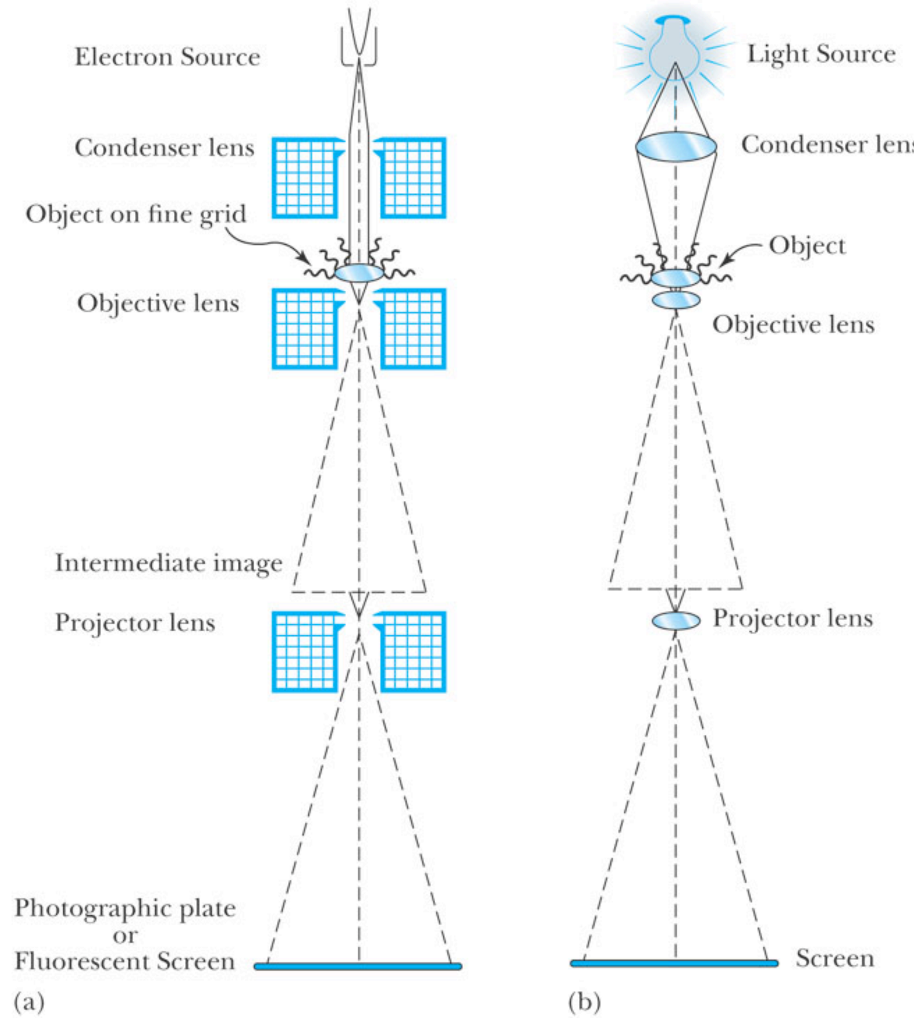


Example crystal diffraction of neutrons.

Electron microscope

The electron microscope is an electronic wave version of the optical transmission microscope.

Magnets serve as focusing lens giving a restoring angular deflection proportional to distance from the axis.



Electron microscope resolution

The angular resolution of an optical system at the “diffraction limit” is determined by the ratio of the wavelength to aperture diameter.

$$\theta_{min} \simeq \frac{\lambda}{d}$$

Optical wavelengths are about 500 nm. The wavelengths of electrons of a few 100 eV are 0.1 nm and the resolution of an electron microscope is potentially 1000 times better than that of an optical microscope.

TEM images

Bacterium

DNA

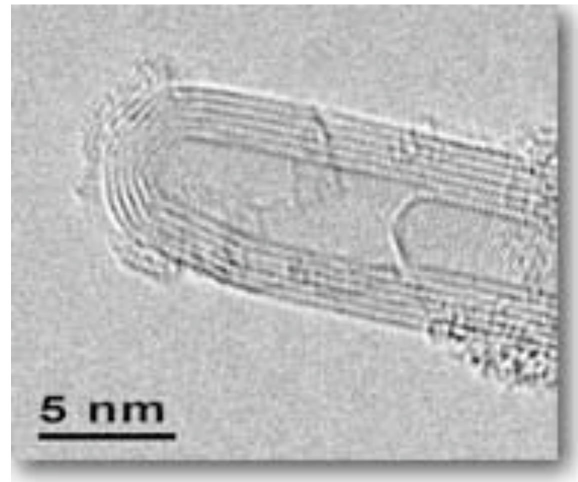
(a)

(b)

TEM images

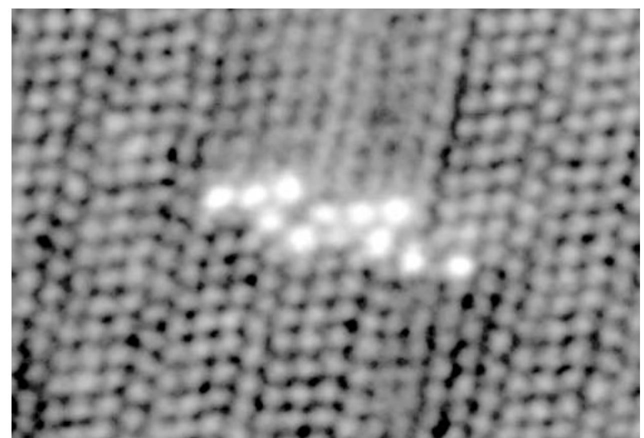
Multi-walled Carbon Nanotube

http://www.ceo.msu.edu/TEMGallery.htm

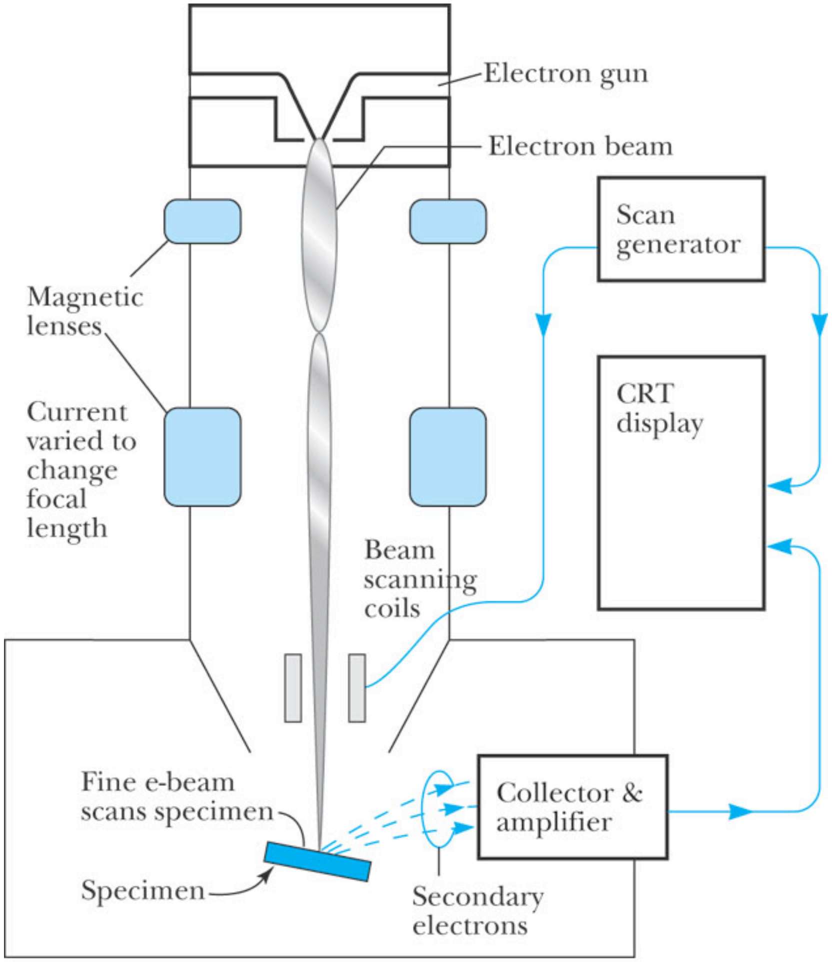


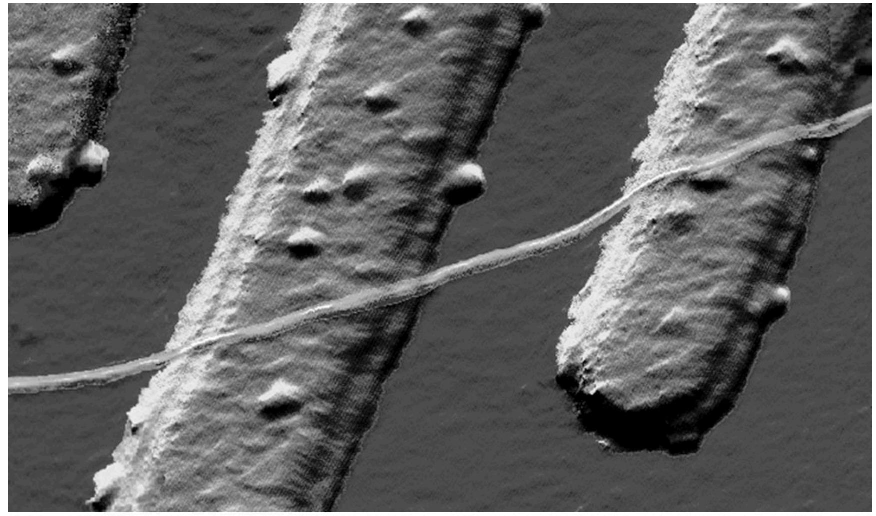
Cluster of Er atoms in SiC matrix.

http://www.jeolusa.com/SERVICESUPPORT/ApplicationsResources/ElectronOptics/ImageGallery/tabid/323/AlbumID/570-5/Default.aspx


Cluster of Er atoms in SiC matrix courtesy of David Muller, Lucent Technologies

Scanning electron microscope

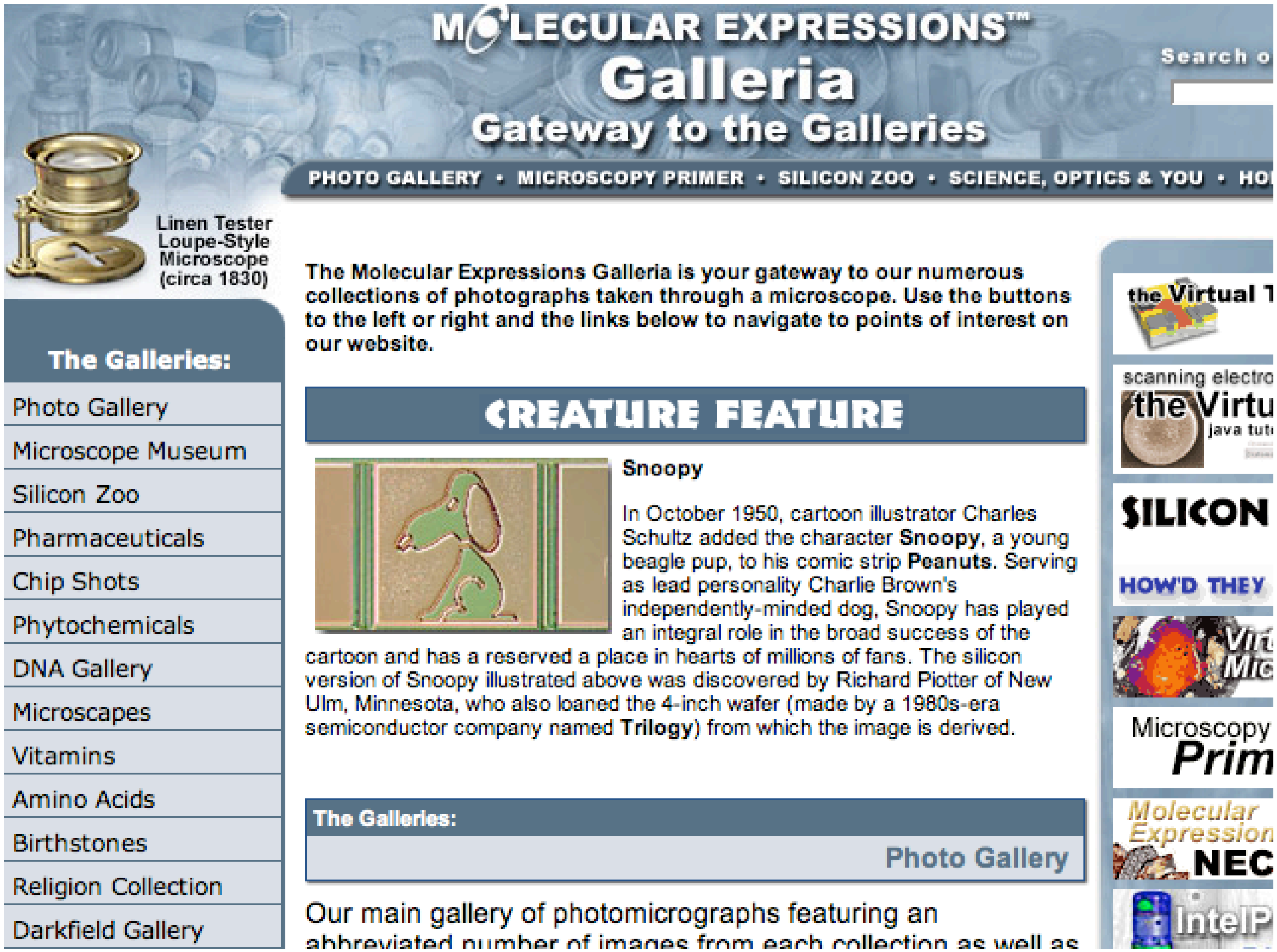




There are many variations!

Optical microscopy still useful!!

http://micro.magnet.fsu.edu/

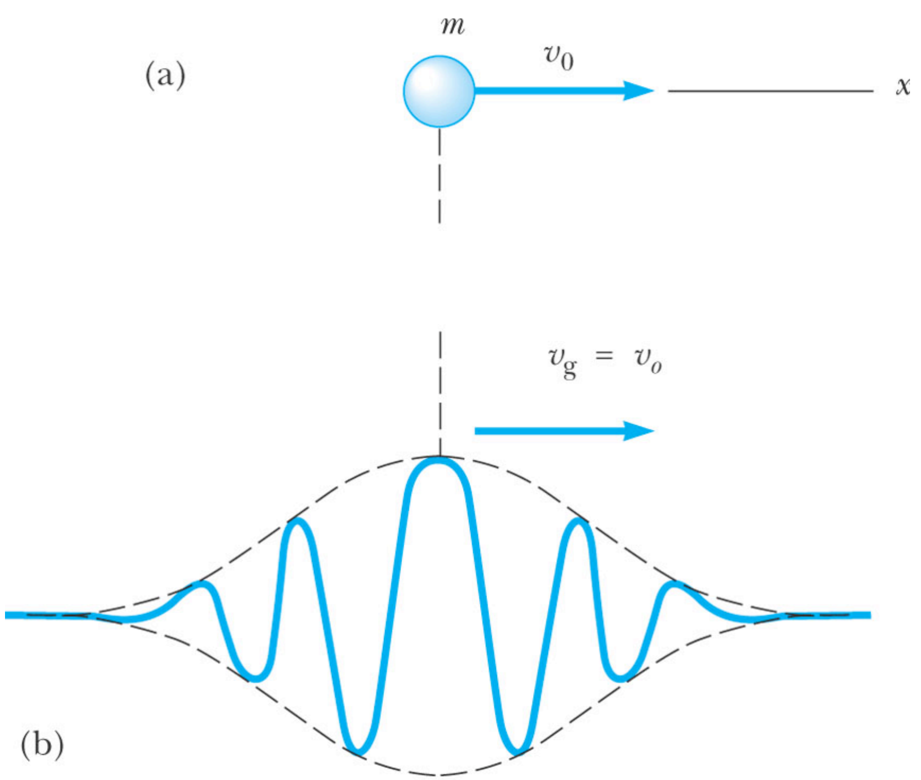


Wave packets

How can a wave emulate a particle?

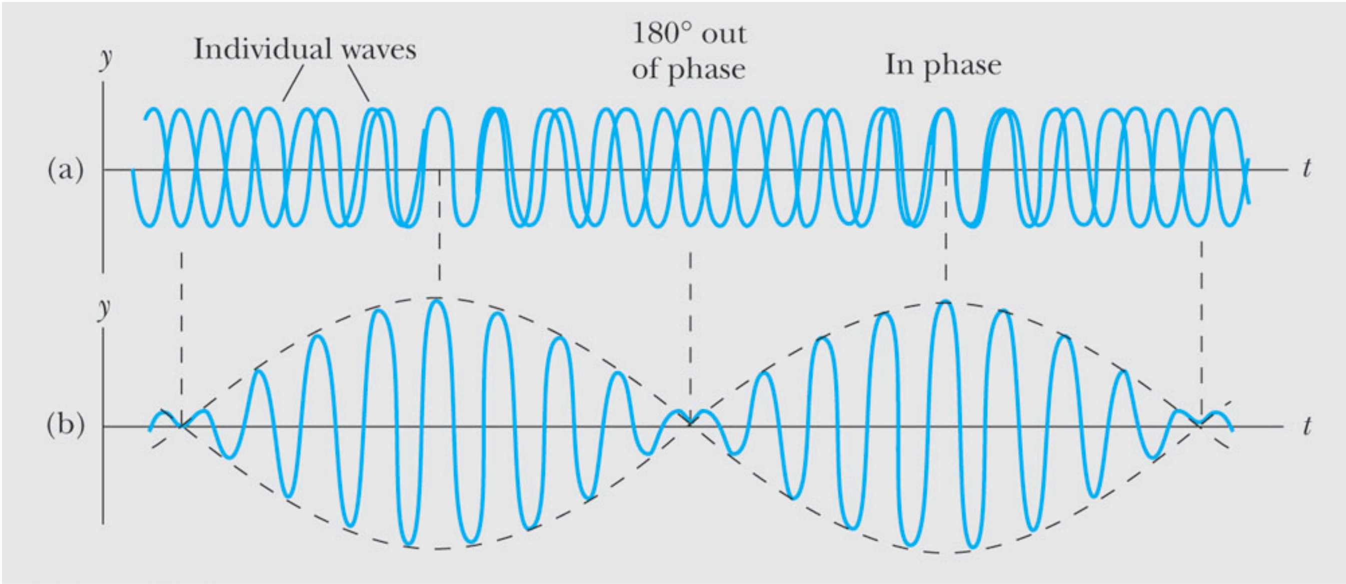
A wave pulse (or packet) is a localized moving wave.

It can be understood as a superposition of harmonic waves.



Superposition

A fundamental property of waves is the principle of superposition. A superposition of two waves of slightly different frequency exhibits beats and localization of a sort.



Wave parameters

A harmonic wave has the form

$$\psi(x,t) = A\cos(kx - \omega t)$$
$$k = \frac{2\pi}{\lambda}\ ;\ \omega = 2\pi f = \frac{2\pi}{T}$$

Here k is called the wave vector and omega is called the angular frequency. This wave travels towards positive x with (phase) velocity:

$$v_{phase} = \frac{\omega}{k}$$

Phase velocity for matter waves

A harmonic non-relativistic matter wave has parameters

$$E = hf = \hbar\omega \;;\; p = \frac{h}{\lambda} = \hbar k$$

$$v_{phase} = \frac{\omega}{k} = \frac{E}{p} = \frac{p^2/(2m)}{p}$$

$$v_{phase} = \frac{p}{2m} = v/2$$

The phase velocity is not fixed as it would be for continuum waves like sound waves which in a given medium travel all at the same speed - it depends on momentum. And the phase velocity is NOT the particle velocity!!

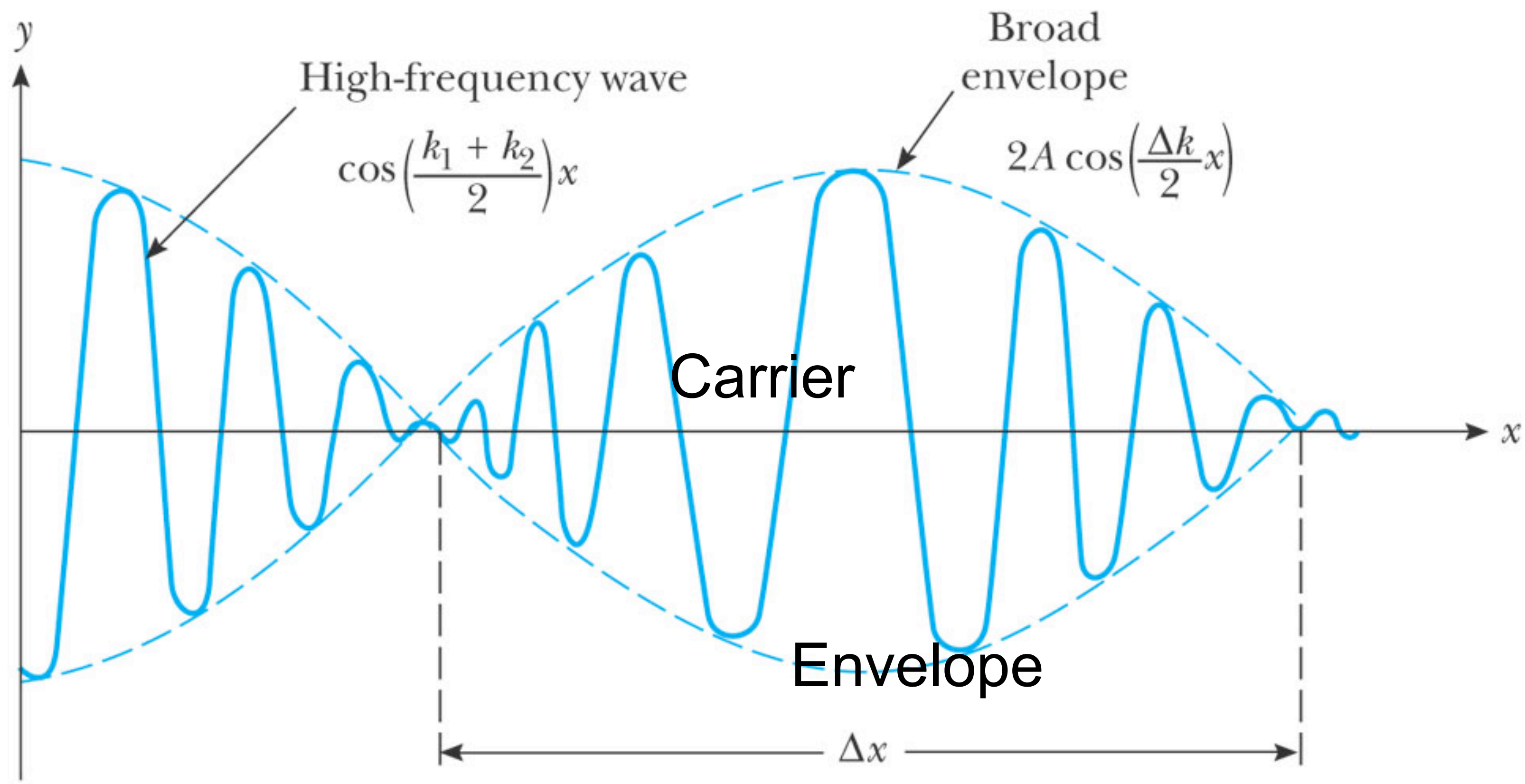
Superposition of two waves

A superposition of two harmonic waves has the form

$$
\begin{aligned}
&\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) = \\
&2\cos(\Delta k x - \Delta\omega t)\cos(\bar{k} x - \bar{\omega} t) \\
&\Delta k = (k_2 - k_1)/2\ ;\ \Delta\omega = (\omega_2 - \omega_2)/2 \\
&\bar{k} = (k_1 + k_2)/2\ ;\ \bar{\omega} = (\omega_1 + \omega_2)/2
\end{aligned}
$$

The sum is an ‘envelope’ wave (depending on the differences of wave vectors and frequencies) times a ‘carrier’ wave (depending on the average of the wave vectors and frequencies).

Superposition of two waves



www.colorado.edu/physics/2000/applets/fourier.html

Group velocity

$$v_{carrier} = \frac{\bar{\omega}}{\bar{k}}$$

$$v_{envelope} = \frac{\Delta\omega}{\Delta k}$$

The carrier and envelope have different velocities in general.

Fourier analysis

Mathematical result:

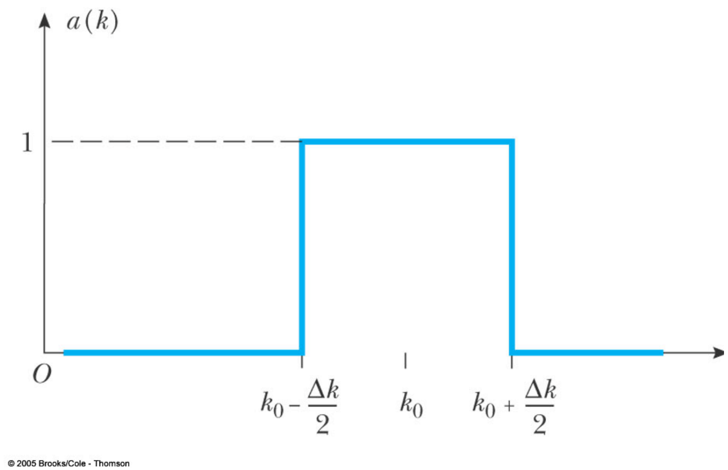
Any (bounded/pulse-like) function can be expressed as an superposition (integral) of plane waves

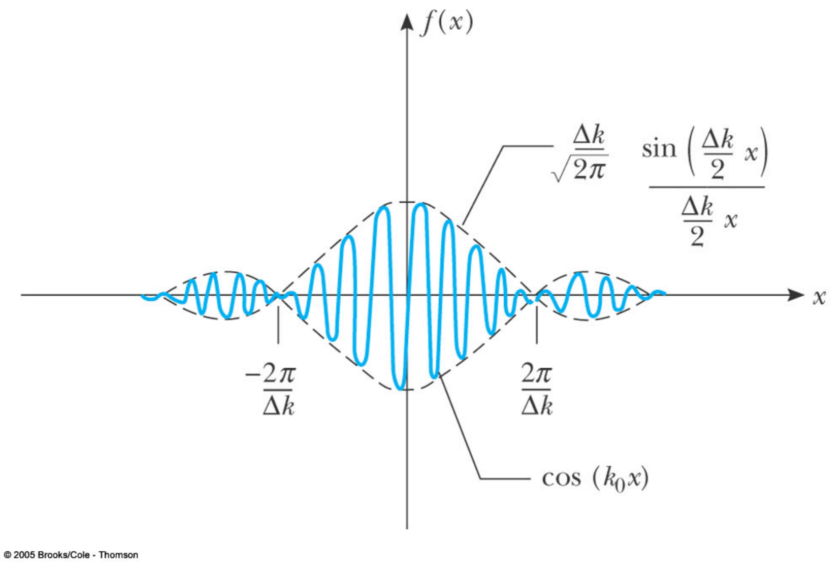
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{ikx} dk$$

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

Here a(k) is the amplitude of the wave e^{ikx}. Given f(x), the second expression tells you a(k). Given a(k), the first expression allows you to construct f(x).

Fourier analysis





Example: a constant amplitude a(k) in a range dk about some k corresponds to a localized pulse.

Motion of a pulse

Suppose we have a known pulse at t=0 and find the amplitude distribution:

$$f(x,t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{ikx} dk$$

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

Each component wave has its own frequency and velocity and we can find the shape at later times as

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{i(kx-\omega(k)t)} dk$$

Dispersion

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{i(kx-\omega(k)t)} dk$$

If the frequency is proportional to wave vector, the pulse is a function of x-vt only, all component waves move at the same speed, and superpose the same way but displaced as time increases and the pulse moves at speed v:

$$\omega = vk$$

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{ik(x-vt)} dk$$

Group velocity

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{i(kx-\omega(k)t)} dk$$

More generally, we can approximate the frequency function by expanding about a value k_0 central to the amplitude distribution:

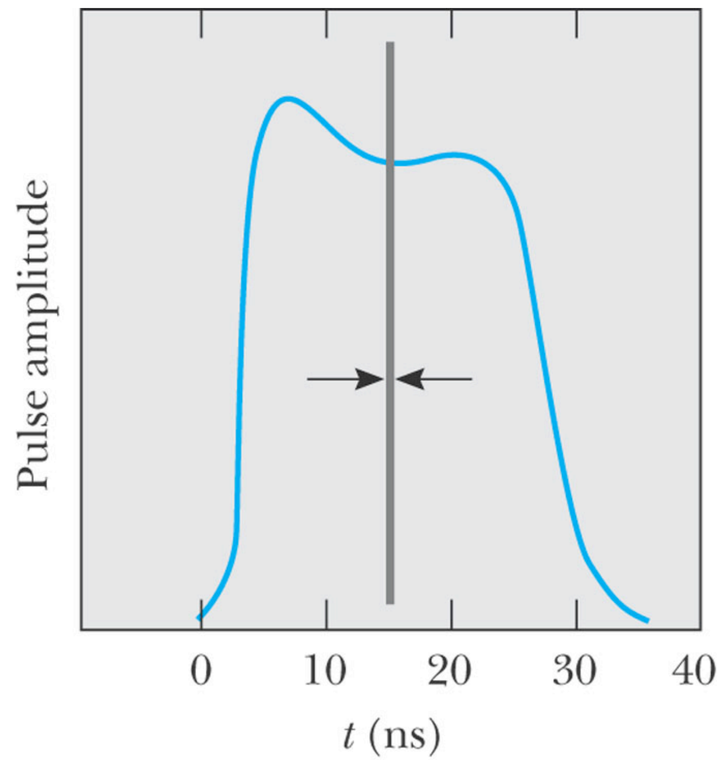
$$\omega = \omega_0 + \frac{d\omega}{dk}(k - k_0) \equiv \omega_0 + v_g(k - k_0)$$
$$f(x,t) = \frac{1}{\sqrt{2\pi}} e^{-i(\omega_0 - v_g k_0)t} \int_{-\infty}^{+\infty} a(k) e^{ik(x - v_g t)} dk$$

The pulse moves at the “group velocity” given by

$$v_{group} = \frac{d\omega}{dk}\Big|_{k_0}$$

Dispersion

If the component waves move at different speeds, the superposition changes with time and the pulse shape changes. A narrow pulse “disperses” - becomes wider with time.



Shape of a 1 ns laser pulse 30 ns after propagating down an optical fiber.

Uncertainty principle

A single frequency wave is spread throughout space. A localized wave pulse has some width dx in space and a range dk of wave vectors with non vanishing amplitude. These are inversely related:

$$\Delta x \Delta k \simeq 1$$

$$\Rightarrow \Delta x \Delta p \simeq \hbar$$

Matter waves: A particle of sharp momentum p is spread throughout space. A particle localized to a region dx has a momentum uncertainty of

$$\Delta p \simeq \frac{\hbar}{\Delta x}$$

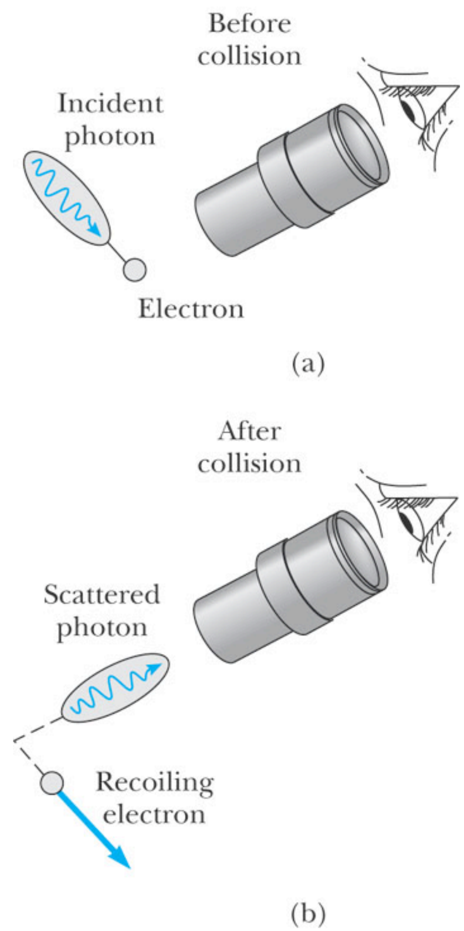
Heisenberg uncertainty principle

Werner Heisenberg provided an interpretation of de Broglie waves and the uncertainty relation in terms of measurement.

Heisenberg uncertainty principle

Consider a prototypical experiment in which light is used to “see” an electron with minimal disturbance.

Suppose for simplicity unit magnfication with a single lens as in the eye. Let f=focal length, D= lens diameter.



Heisenberg uncertainty principle

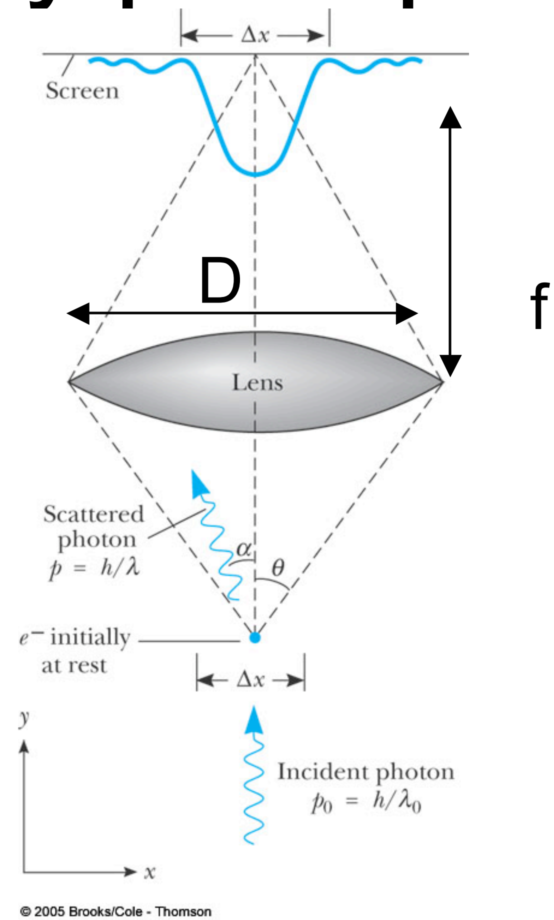
Because light is a wave,
diffraction implies an image size
(f=focal length, D= lens diameter)

$$\Delta x \simeq \delta\theta f \simeq \frac{\lambda}{D} f$$

Individual photons have momentum
in a range intercepted by the lens or

$$\Delta p_x \simeq (2\theta)p = \frac{D}{f}\frac{h}{\lambda}$$

In the single quantum scattering,
the photons transfer this uncertain
momentum to the electron and so..



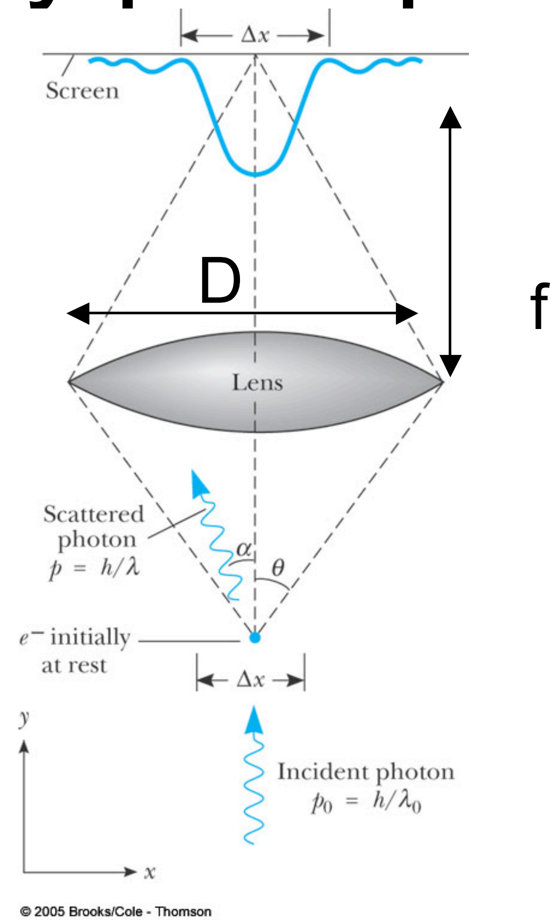
$$\Delta p_x \Delta x \simeq h$$

Heisenberg uncertainty principle

Interpretation:

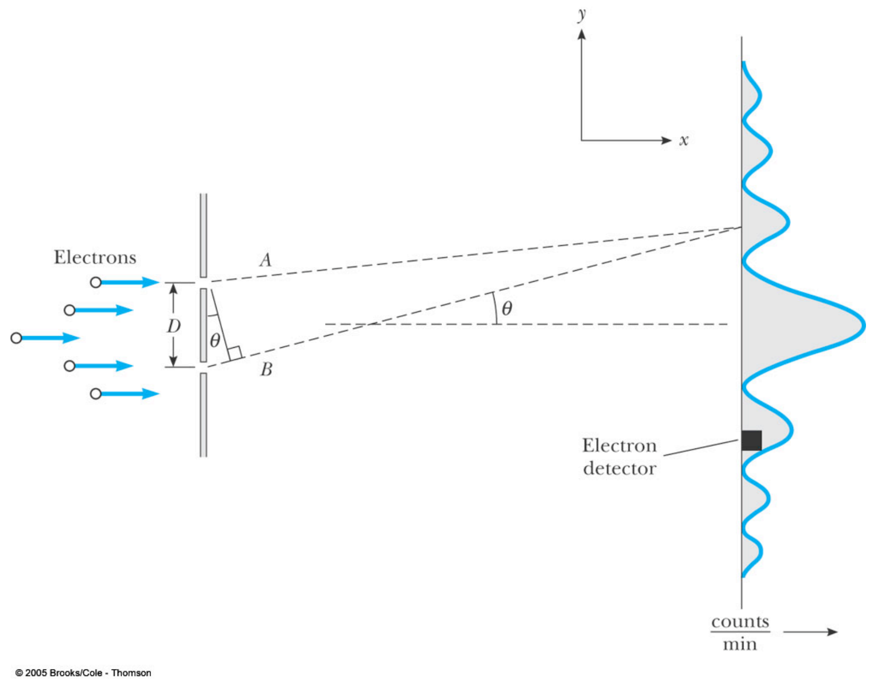
In so far as all matter and light has the dual wave/particle character, a measurement that determines the position of a particle to within a range dx necessarily implies it imparts an uncertainty in momentum dp=h/dx.

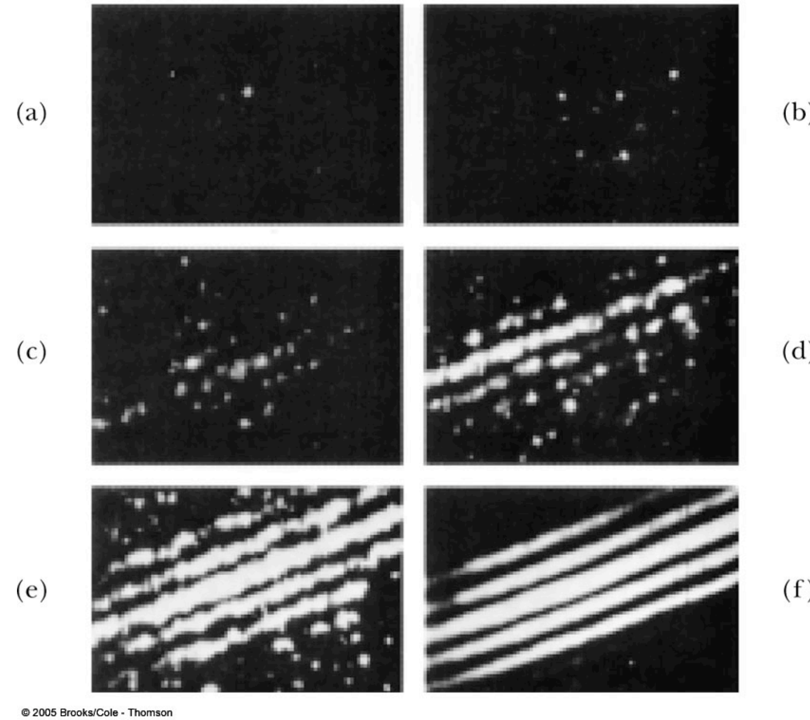
No particle has simultaneously a unique position and momentum.



$$\Delta p_x \Delta x \simeq h$$

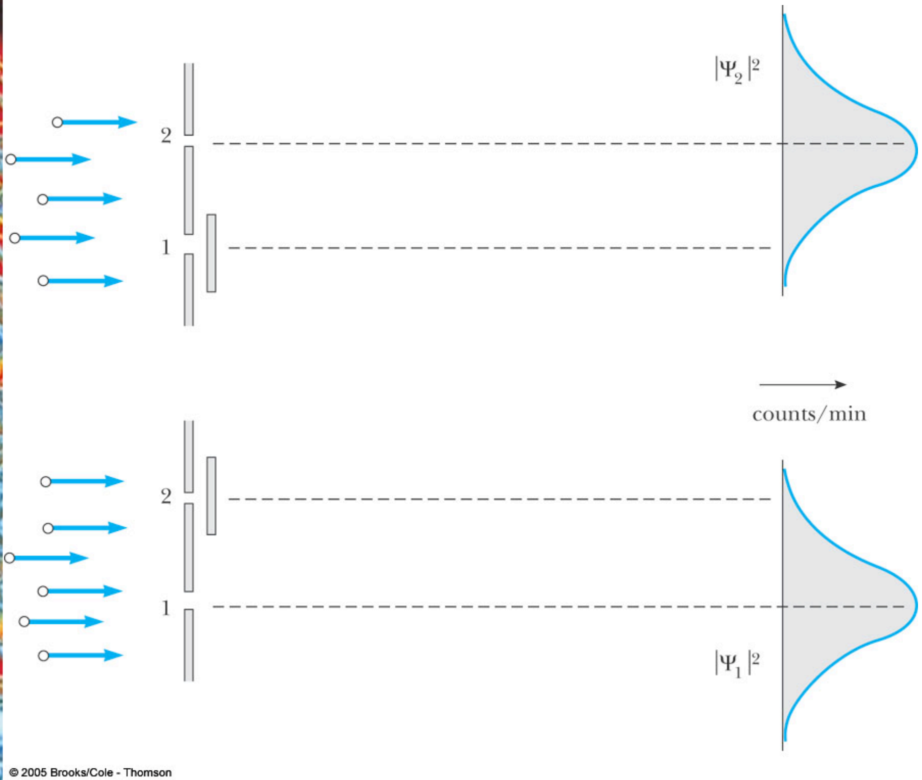
Two slit experiment

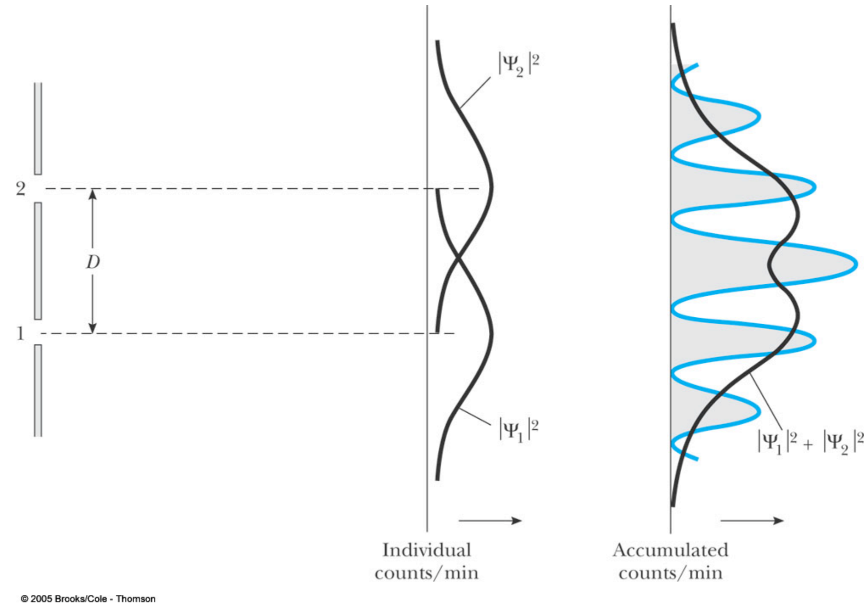




At low beam intensity, one observes the quantum nature of matter and light - single particle events assembling randomly to form the wave interference pattern.

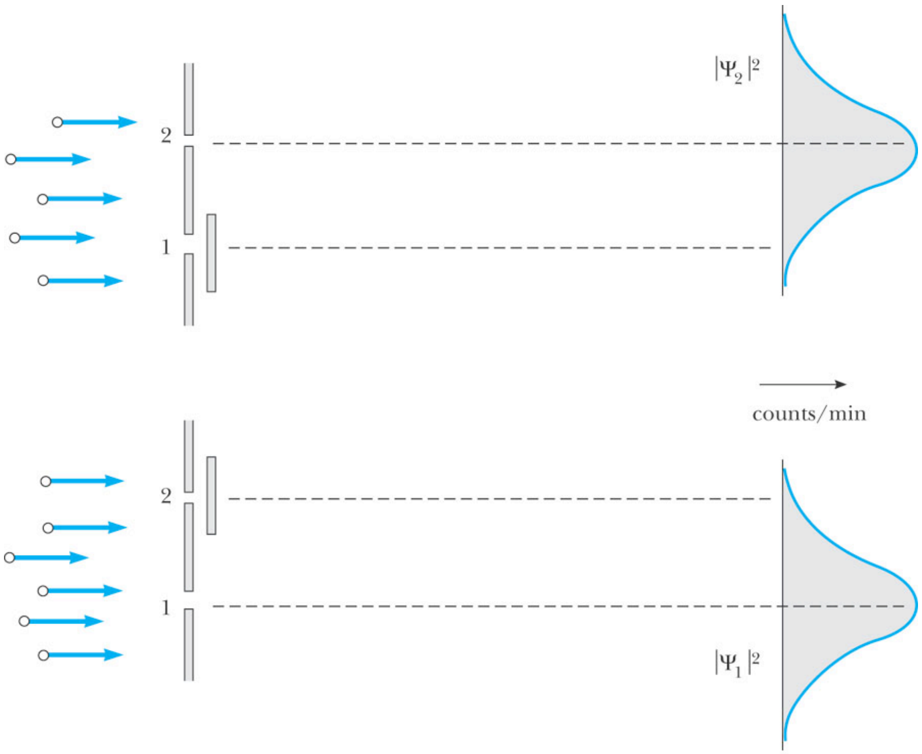
Two slit interference

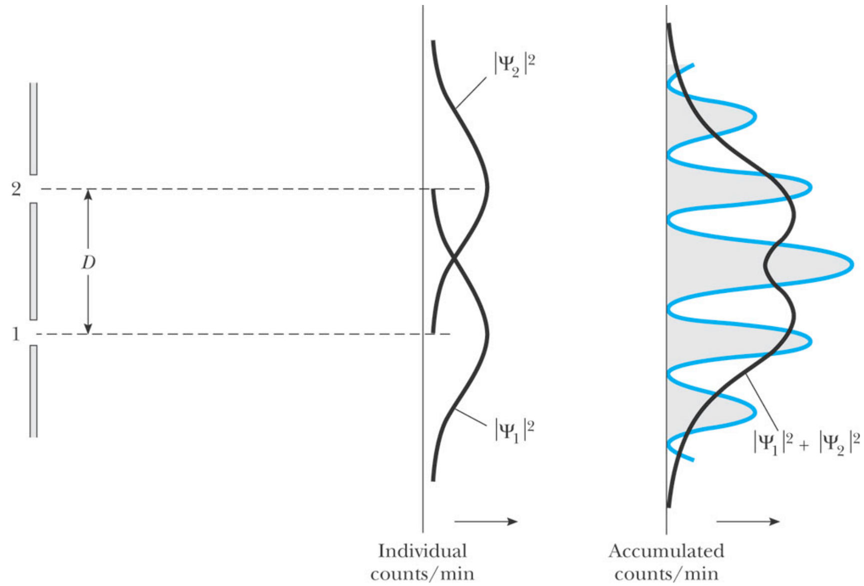




If one slit or the other is blocked, one observes a single slit diffraction pattern. With both slits open, the pattern is NOT the sum. Instead a two slit interference pattern.

Two slit weirdness

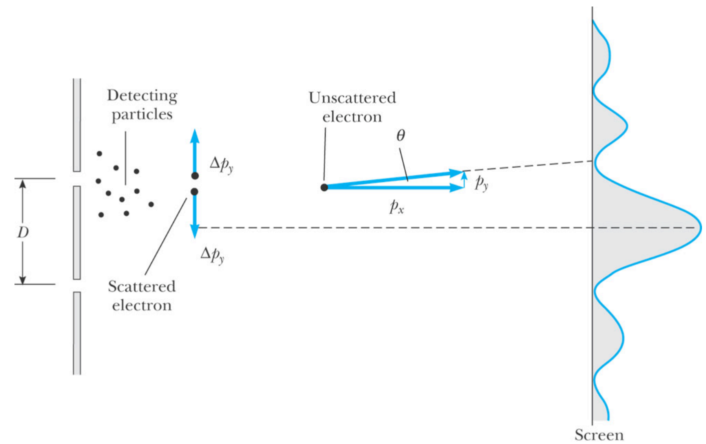




Starting from one slit open, opening the other yields at some angles no particles where there were some previously (destructive interference) and more than double the number at other angles (constructive interference).

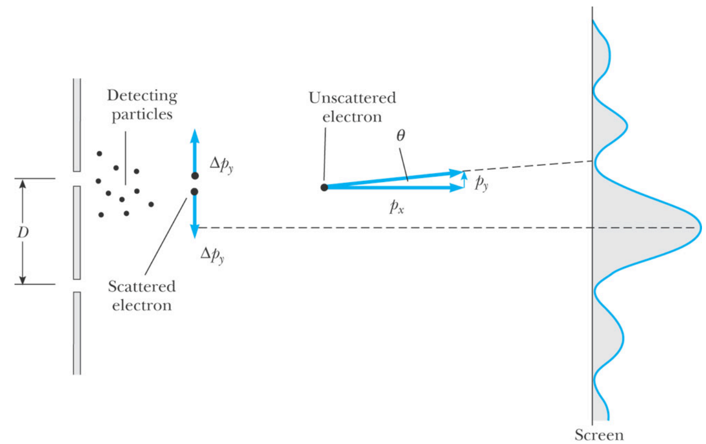
A tricky 2 slit experiment

It seems particle explore both slits simultaneously. Suppose we place some electrons behind one slit to see which slit each particle is going through.



A tricky 2 slit experiment

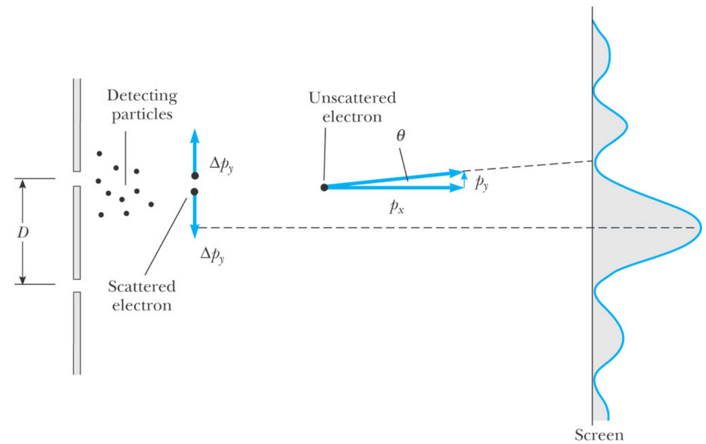
The target particles must have some dy<<D and by the uncertainty principle $dp_y \sim h/dy >> h/D$
This momentum uncertainty is transferred to the beam particles and will wash out the interference pattern!



This experiment is different and gives different results! The uncertainty principle still rules!

Lesson from 2 slit experiments

It is not possible to establish that a particle follows a trajectory in the classical sense except within the limitations prescribed by the uncertainty principle.



Time-energy uncertainty

Superposition of waves with wavevectors in a range dk =dp/hbar gives a pulse in space of minimal width dx~1/dk

$$\Delta p_x \Delta x \simeq h$$

Similarly, superposition of waves with frequencies in a range d omega = dE/hbar gives a pulse in time of minimal width dT~1/d omega

$$\Delta E \Delta T \simeq h$$

Time-energy uncertainty

$$\Delta E \Delta T \simeq h$$

Interpretation: The energy of a particle that is observed over a limited time range dT is minimally uncertain by dE.

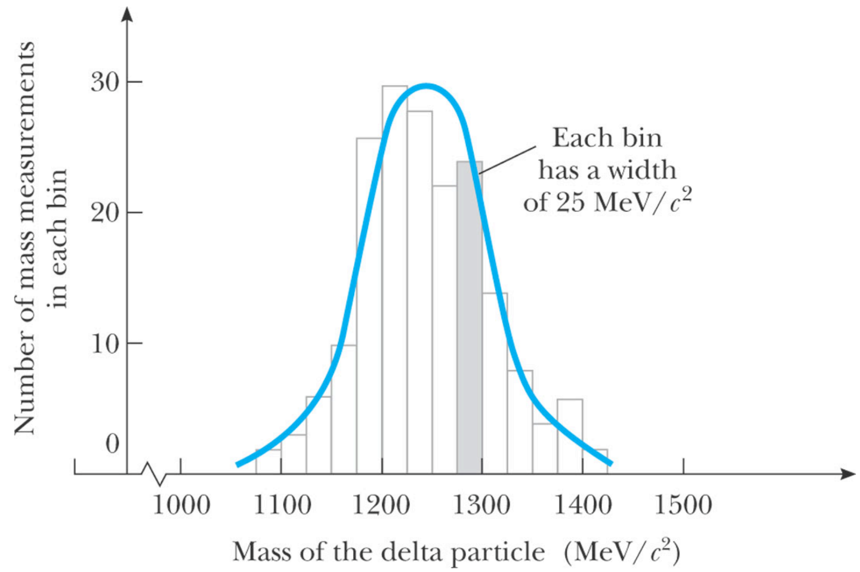
Example: Unstable particles and electrons in excited atomic states are observed to decay exponentially in time

$$P(t) = e^{-t/\tau}$$

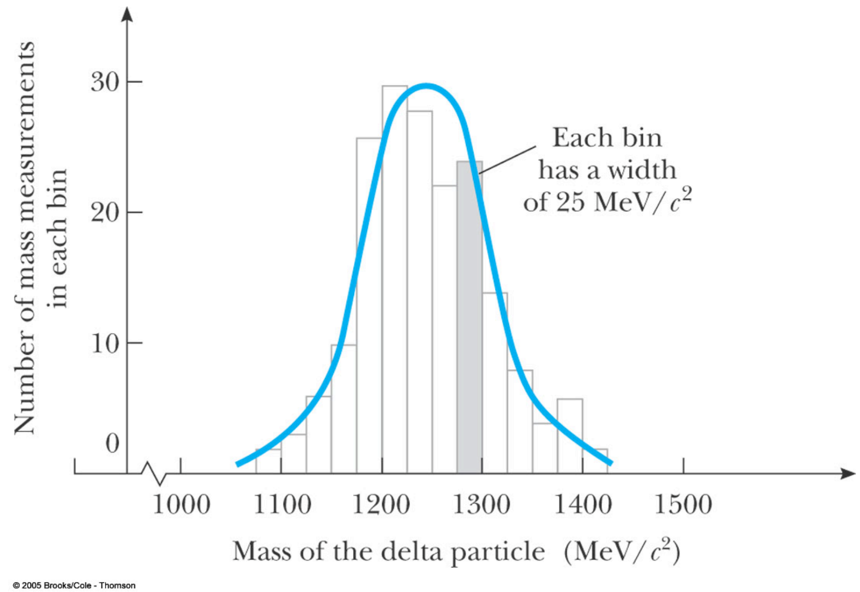
The uncertainty in their energy is $\Delta E \simeq h/\tau$

Time-energy uncertainty

Example:
The total energy and momentum of decay products is used to reconstruct tot total energy, momentum and rest mass of a particle. The distribution of rest energy has a width of 100 MeV. What is the lifetime?



Time-energy uncertainty



$$\tau = \frac{h}{\Delta E} = \frac{4e - 15\ eV - s}{100e6\ eV} \sim 10^{-23} s$$